The Linear Algebra of Economic Geography Models

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Abstract

We provide sufficient statistics for nominal and real wage exposure to productivity shocks in a constant elasticity economic geography model. These exposure measures summarize the first-order general equilibrium elasticity of nominal and real wages in each location with respect to productivity shocks in all locations. They are readily computed using commonly-available trade data and the values of trade and migration elasticities. They have an intuitive interpretation in terms of underlying economic mechanisms. Computing these measures for all bilateral pairs of locations involves a single matrix inversion and therefore remains computational efficient even with an extremely high-dimensional state space. These sufficient statistics provide theory-consistent measures of locations' exposure to productivity shocks for use in further economic and statistical analysis.

Keywords: economic geography, trade, migration

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1 Introduction

A recent advance in spatial economics has been the development of quantitative spatial models. These models are sufficiently rich to capture first-order features of the data, such as many heterogeneous locations, connected by a rich geography of trade costs. Yet these models remain sufficiently tractable as to permit an analytical characterization of the general equilibrium. They are also relatively parsimoniously specified, with a small number of equilibrium relationships and parameters to be estimated, thereby permitting transparent interpretation of results. They thus provide a platform undertaking a wide range of counterfactuals, including for the impact of productivity shocks in individual locations or transport infrastructure improvements between pairs of locations.[1]

We show that comparative statics for productivity shocks in a constant elasticity economic geography model can be represented using a friend-enemy matrix that summarizes each location's exposure to productivity shocks in all locations. Our approach involves two key steps. We first stack the conditions for the first-order general equilibrium effects of productivity shocks in matrix form. We next invert this matrix system of equations to recover the full bilateral network of each location's exposure to productivity shocks in all locations.

This friend-enemy matrix representation has three attractive properties. First, it provides closed-form sufficient statistics for the impact of productivity shocks in terms of observed trade share matrices and parameters. Second, it is computationally efficient, allowing comparative statics to be computed almost instantaneously, even for extremely high-dimensional state spaces. Third, it has a simple and intuitive interpretation in terms of underlying economic mechanisms in the model. The resulting friend-enemy exposure measures provide theory-consistent measures of locations' exposure to productivity shocks that can be used as inputs in further economic and statistical analysis.

Our paper is related to two main strands of research. First, our work builds on quantitative

[1] For surveys of this literature on quantitative spatial models, see Redding and Rossi-Hansberg (2017), Redding (2022, 2023) and Allen and Arkolakis (2023).

models of international trade between countries. In a class of constant elasticity trade models, Arkolakis et al. (2012) show that a country's share of expenditure on itself and the elasticity of trade flows to trade costs are sufficient statistics for the welfare gains from trade. In trade and economic geography models characterized by a gravity equation, Allen et al. (2020) show that the existence, uniqueness and counterfactual predictions of these models depend only on observed data and demand and supply elasticities. In trade and production networks with distortions, Baqaee and Farhi (2019) derive microeconomic sufficient statistics for the general equilibrium response of variables to productivity and trade cost shocks. Manipulating the first-order conditions for general equilibrium in constant elasticity trade models, Kleinman et al. (2020) derive friend-enemy exposure measures of the elasticity of real income to foreign productivity growth, and provide evidence that changes in real income exposure cause changes in bilateral political alignment.

Second, our work is related to research on economic geography following Krugman (1991), including Helpman (1998), Fujita et al. (1999), Redding and Sturm (2008) and Redding (2016). In contrast to the literature on international trade, this research on economic geography allows for labor mobility across locations. Considering a class of constant elasticity economic geography models, Allen and Arkolakis (2014) provide conditions for the existence and uniqueness of equilibrium, while Adão et al. (2019) provide sufficient statistics for the differential and aggregate impact of trade shocks on local labor markets. Our main contribution relative to this research is to manipulate the first-order general equilibrium conditions in these constant elasticity economic geography models to derive a friend-enemy representation of each location's exposure to productivity shocks in all locations, which is analogous to but differs from the corresponding friend-enemy representation in international trade models.[2]

The remainder of the paper proceeds as follows. Section 2 introduces our theoretical framework. Section 3 derives our friend-enemy exposure measures. Section 4 concludes. Throughout,

[2]We focus on static economic geography models, and hence abstract from dynamics due to migration (as in Caliendo et al. 2019) or migration and capital accumulation (as in Kleinman et al. 2023). We also abstract from input-output linkages (as in Liu 2019 and Liu and Tsyvinski 2023).

we use bold math font to denote vectors or matrices. The derivation of all theoretical results is contained in an accompanying Online Appendix.

2 Theoretical Model

The world economy consists of a set of locations indexed by $i, n \in \{1, \ldots, N\}$. The economy has an exogenous supply of workers that we normalize to one ($\bar{\ell} = 1$). Each worker is endowed with one unit of labor that is supplied inelastically. Workers are perfectly mobile across locations, but have idiosyncratic preferences for each location.

2.1 Consumer Preferences

The preferences of worker ν who chooses to live in location n are characterized by the following indirect utility function:

$$u_n(\nu) = \frac{b_n \epsilon_n(\nu) w_n}{p_n}, \tag{1}$$

where w_n is the wage, p_n is the consumption goods price index; b_n captures amenities that are common for all workers (such as climate and scenic views); and $\epsilon_n(\nu)$ is an idiosyncratic amenity draw that is specific to each worker ν and location n. The consumption goods price index is assumed to take the following constant elasticity of substitution (CES) form:

$$p_n = \left[\sum_{i=1}^{N} p_{ni}^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \qquad \sigma > 1. \tag{2}$$

Idiosyncratic amenities are drawn independently for each worker and location from the following Fréchet distribution:

$$F(\epsilon) = \exp\left(-\epsilon^{-\kappa}\right), \qquad \kappa > 1, \tag{3}$$

where we normalize the scale parameter to one, because it enters the model isomorphically to b_n; the shape parameter $\kappa > 1$ regulates the dispersion of idiosyncratic amenities, and determines the migration elasticity that captures the responsiveness of population shares to real wages.

2.2 Production Technology

Goods are produced with labor according to a constant returns to scale production technology and under conditions of perfect competition. These goods can be traded between locations subject to iceberg variable costs of trade, such that $\tau_{ni} \geq 1$ units must be shipped from location i to location n in order for one unit to arrive. Therefore, the cost to the consumer in location n of purchasing the good produced by location i is:

$$p_{ni} = \frac{\tau_{ni} w_i}{z_i}, \tag{4}$$

where z_i captures productivity in location i and iceberg variable trade costs satisfy $\tau_{ni} > 1$ for $n \neq i$ and $\tau_{nn} = 1$.

For comparability with the international trade literature, we focus on the case in which productivity (z_i) is exogenous. Nevertheless, it is straightforward to introduce agglomeration economies, whereby productivity in each location is increasing in its own population, or the population of surrounding locations.

2.3 General Equilibrium

General equilibrium can be referenced by the vectors of wages and population shares in each location $\{w_n, \ell_n\}$. The $2 \times N$ values of wages and population shares are determined by the $2 \times N$ equilibrium conditions from goods market clearing and population mobility. Goods market clearing requires that income in each location equals expenditure on the goods produced by that location:

$$w_i \ell_i = \sum_{n=1}^{N} s_{ni} w_n \ell_n, \tag{5}$$

where s_{ni} is the share of expenditure of importer n on exporter i. From CES demand (2) and the production technology (4), this expenditure share is given by:

$$s_{ni} = \frac{\left(\tau_{ni} w_i / z_i\right)^{-\theta}}{\sum_{m=1}^{N} \left(\tau_{nm} w_m / z_m\right)^{-\theta}}, \tag{6}$$

where $\theta \equiv \sigma - 1$ is the trade elasticity.

We choose the total income of all locations as the numeraire:

$$\sum_{i=1}^{N} q_i = 1, \tag{7}$$

where $q_i \equiv w_i \ell_i$ is the nominal income of location i.

Using the properties of the Fréchet distribution (3), the probability a worker chooses to live in location n is:

$$\ell_n = \frac{(b_n w_n / p_n)^{\kappa}}{\sum_{h=1}^{N} (b_h w_h / p_h)^{\kappa}}, \tag{8}$$

and expected utility conditional on choosing to live in a location is the same across all locations and given by:

$$\bar{u} = \Gamma\left(\frac{\kappa - 1}{\kappa}\right) \left[\sum_{h=1}^{N} (b_h w_h / p_h)^{\kappa}\right]^{\frac{1}{\kappa}}, \tag{9}$$

where $\Gamma(\cdot)$ is the Gamma function.

Intuitively, each location faces an upward-sloping supply function for workers in equation (8), such that it has to offer a higher real wage relative to other locations in order to attract a larger share of the population. Nevertheless, expected utility is still equalized across all locations in equation (9), because locations that offer higher real wages attract workers with lower realizations for idiosyncratic amenities. With a Fréchet distribution for idiosyncratic amenities, the higher real wage is exactly offset by this composition effect from lower average amenities, such that expected utility conditional on choosing a location is the same across all locations.

Given our assumption of exogenous productivity, there are no agglomeration forces in the model. Therefore, the dispersion force from worker idiosyncratic preferences ensures the existence of a unique equilibrium distribution of wages and population shares $\{w_n, \ell_n\}$ across locations.

3 Friend-Enemy Exposure Measures

We consider small productivity shocks, holding constant amenities ($\mathrm{d}\ln b_i = 0$), bilateral trade costs ($\mathrm{d}\ln \tau_{ni} = 0$), and the total population of the economy ($\mathrm{d}\ln \bar{\ell} = 0$). Totally differentiating the goods market clearing condition (5), the general equilibrium change in location income can be represented to first-order as:

$$\mathrm{d}\ln\mathbf{w} + \mathrm{d}\ln\boldsymbol{\ell} = \mathbf{T}\left(\mathrm{d}\ln\mathbf{w} + \mathrm{d}\ln\boldsymbol{\ell}\right) + \theta\left(\mathbf{TS} - \mathbf{I}\right)\left(\mathrm{d}\ln\mathbf{w} - \mathrm{d}\ln\mathbf{z}\right), \tag{10}$$

where $\boldsymbol{S}$ is a matrix with elements S_{ni} for the share of importer n's expenditure on exporter i; $\boldsymbol{T}$ is a matrix with elements $T_{in} = S_{ni}w_n\ell_n/\left(w_i\ell_i\right)$ equal to the share of exporter i's income from importer n.

Intuitively, the change in the total income of a location on the left-hand side of this equation (including both changes in wages and changes in population) depends on a *market-size effect* (first term on the right-hand side) and a *cross-substitution effect* (second term on the right-hand side). The *market-size effect* captures the fact that an increase in the income of market n on the right-hand side raises the income of location i on the left-hand side by an amount that is determined by the share of location i's income from market n (T_{in}).

The *cross-substitution effect* depends on the product of the income and expenditure share matrices ($\boldsymbol{TS} - \boldsymbol{I}$). For $i \neq n$, the sum $\sum_{h=1}^{N} t_{ih}s_{hn}$ captures the overall competitive exposure of location i to location n through each of their common markets h, weighted by the importance of market h for country i's income (t_{ih}). As the competitiveness of location n increases, as measured by a decline in its wage relative to its productivity ($\mathrm{d}\ln w_n - \mathrm{d}\ln z_n$), consumers in all markets h substitute towards location n and away from other locations $i \neq n$. This substitution effect reduces income in location i and raises it in location n. With a constant elasticity import demand system, the magnitude of this cross-substitution effect in market h depends on the trade elasticity (θ) and the share of expenditure in market h on the goods produced by location n (s_{hn}): consumers in market h increase the expenditure share on country n by $(1 - s_{hn})$ and lower the expenditure share on country i by s_{hn}.

This goods market clearing condition (10) takes a similar form as in a constant elasticity international trade model. The key difference is that the population shares (ℓ_n) are endogenous and affect the income of each location on both the left and right-hand sides of the equation.

Totally differentiating the location choice probabilities (8), the general equilibrium change in these population shares can be represented to first-order as:

$$\mathrm{d}\ln \boldsymbol{\ell} = (\mathbf{I} - \mathbf{1}\boldsymbol{\ell}')\mathrm{d}\ln \boldsymbol{\ell} = \kappa(\mathbf{I} - \mathbf{1}\boldsymbol{\ell}')\left[\mathrm{d}\ln \mathbf{w} - \mathbf{S}\left(\mathrm{d}\ln \mathbf{w} - \mathrm{d}\ln \mathbf{z}\right)\right], \tag{11}$$

where the first equality reflects the fact that total population is constant.

Intuitively, the population share of a location is increasing in its wage (the first term inside the square parentheses on the right-hand side) and decreasing in its consumption price index (the second term inside these square parentheses). This change in the consumption price index in turn equals the expenditure share-weighted average of changes in wages relative to productivities in all locations.

Totally differentiating expected utility (9), the general equilibrium change in the common level of utility across all locations can be represented to first-order as the population-share weighted average of the change in the real wage in each location:

$$\mathrm{d}\ln \bar{u} = \boldsymbol{\ell}'\left[\mathrm{d}\ln \mathbf{w} - \mathbf{S}\left(\mathrm{d}\ln \mathbf{w} - \mathrm{d}\ln \mathbf{z}\right)\right], \tag{12}$$

where the change in the real wage inside the square parentheses equals the change in the nominal wage minus the change in the consumption price index.

In this economic geography model, productivity shocks that change the distribution of real wages induce a reallocation of population across locations through equation (11), which changes income in each location ($w_i\ell_i$), and hence feeds back to influence wages in each location through the market-size effect in the goods market clearing condition (10). In the new equilibrium, the changes in real wages and population shares must be exactly such that all locations experience the same change in expected utility in equation (12).

We now manipulate the goods market clearing condition (10), population shares (11) and expected utility (12) to derive our nominal and real wage exposure measures. Using our choice of numeraire (7), we first re-write the goods market clearing condition (10) as follows:

$$\left(\mathbf{I} + \mathbf{Q}\right)\left(\mathrm{d}\ln \mathbf{w} + \mathrm{d}\ln \boldsymbol{\ell}\right) = \mathbf{T}\left(\mathrm{d}\ln \mathbf{w} + \mathrm{d}\ln \boldsymbol{\ell}\right) + \theta\left(\mathbf{TS} - \mathbf{I}\right)\left(\mathrm{d}\ln \mathbf{w} - \mathrm{d}\ln \mathbf{z}\right), \tag{13}$$

where $\mathbf{Q}$ is a $N \times N$ matrix with the nominal income row vector $\boldsymbol{q}'$ stacked N times, and our choice of numeraire (7) implies $\mathbf{Q}\left(\mathrm{d}\ln\mathbf{w}+\mathrm{d}\ln\boldsymbol{\ell}\right)=0$. We include this term for our choice of numeraire to determine the units in which nominal wages are measured.

Substituting the changes in population shares (11) into goods market clearing (10), and using the change in the common level of expected utility (12), we obtain the following closed-form expression for the elasticity of wages in each location with respect to productivity shocks in all locations:

$$\mathrm{d}\ln\boldsymbol{w}=\mathbf{W}\mathrm{d}\ln\boldsymbol{z} \tag{14}$$

where $\mathbf{W}$ is our friend-enemy matrix of bilateral wage exposure to productivity shocks in all locations:

$$\mathbf{W}\equiv\begin{bmatrix} -\left[\left(\frac{1+\kappa+\theta}{1+\kappa}\right)\mathbf{I}-\mathbf{V}\right]^{-1} \\ \times\left[\left(\frac{\theta-\kappa}{1+\kappa}\right)\mathbf{TS}-\left(\frac{\theta}{1+\kappa}\right)\mathbf{I}+\frac{\kappa}{1+\kappa}\mathbf{S}\right]\end{bmatrix} \tag{15}$$

$$\mathbf{V}\equiv\mathbf{T}+\left(\frac{\theta-\kappa}{1+\kappa}\right)\mathbf{TS}+\frac{\kappa}{1+\kappa}\mathbf{S}-\mathbf{Q}. \tag{16}$$

and the presence of $\mathbf{Q}$ ensures that the matrix $\left[\left(\frac{1+\kappa+\theta}{1+\kappa}\right)\mathbf{I}-\mathbf{V}\right]$ is invertible.[3]

We can also compute an analogous measure of real wage exposure to productivity shocks in all locations ($\mathbf{U}$), such that the common change in expected utility (12) across all locations can be written as:

$$\mathrm{d}\ln\bar{u}=\boldsymbol{\ell}'\mathbf{U}\mathrm{d}\ln\boldsymbol{z} \tag{17}$$

where real income exposure ($\mathbf{U}$) is:

$$\mathbf{U}\equiv\left[\left(\mathbf{I}-\mathbf{S}\right)\mathbf{W}+\mathbf{S}\right], \tag{18}$$

and is invariant to the choice of numeraire.

We thus obtain sufficient statistics for the exposure of nominal and real wages in each location to productivity shocks in all locations. These sufficient statistics depend solely on the

[3]The expenditure and income shares both sum to one, which implies that the rows and columns of $\mathbf{S}$ and $\mathbf{TS}$ are not linearly independent. Therefore, without the inclusion of the term in $\mathbf{Q}$, the matrix $\left[\left(\frac{1+\kappa+\theta}{1+\kappa}\right)\mathbf{I}-\mathbf{V}\right]$ is not invertible. Economically, this reflects the fact that expenditure and income shares are homogeneous of degree zero in wages, such that that level of wage exposure cannot be recovered from these expenditure and income shares without a choice of numeraire.

observed expenditure share ($\mathbf{S}$) and income share ($\mathbf{T}$) matrices and the two parameters of the trade elasticity (θ) and migration elasticity (κ).

4 Conclusions

We provide sufficient statistics for nominal and real wage exposure to productivity shocks in a constant elasticity economic geography model. These exposure measures summarize the first-order general equilibrium elasticity of nominal and real wages in each location with respect to productivity shocks in all locations. They are readily computed using commonly-available trade data and the values of trade and migration elasticities. They have an intuitive interpretation in terms of underlying economic mechanisms in the model. Computing these measures for all bilateral pairs of locations involves a single matrix inversion and therefore remains computational efficient even in environments with an extremely high-dimensional state space. These sufficient statistics provide theory-consistent measures of locations' exposure to productivity shocks for use in further economic and statistical analysis.

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